数理解析研究所講究録
1091 巻 1999 年 233-240

On some additive problems with primes and almost-primes

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§1. Introduction and statement of the result.

In 1937 I. M. Vinogradov [11] solved the ternary Goldbach problem proving that for every sufficiently large odd integer N the equation

$$p_1 + p_2 + p_3 = N \tag{1}$$

has solutions in prime numbers p_1, p_2, p_3.

Two years later van der Corput [3] used the method of I. M. Vinogradov and established the existence of infinitely many arithmetic progressions of three different primes. A corresponding result for progressions of four or more primes has not been proved yet. In 1981, however, D. R. Heath-Brown [4] proved that there exist infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is P_2 (as usual P_r denotes an integer with no more than r prime factors, counted according to multiplicity).

Another famous and still unsolved number theory problem is the so-called *prime twins* conjecture, which asserts that there exist infinitely many primes p, such that $p+2$ is also a prime. The most important achievement in studying this problem is due to Chen [2]. In 1973 he proved that there exist infinitely many primes p, such that $p+2$ is P_2.

In 1997 D. I. Tolev and the author [8] applied the Hardy–Littlewood circle method and the Bombieri–Vinogradov theorem as well as some arguments belonging to H. Mikawa, and proved that there exist infinitely many non-trivial arithmetic progressions of three primes, such that for two of them, p_1 and p_2, say, both the numbers $p_1 + 2$, $p_2 + 2$ are almost–primes.

Later D. I. Tolev [9] obtained an extension of the above result by applying the vector sieve developed by Iwaniec [5] and used also by Brüdern and Fouvry [1]. He established that the equation

$$p_1 + p_2 = 2p_3$$

has infinitely many solutions in different primes p_1, p_2, p_3, such that $p_1 + 2 = P_5$, $p_2 + 2 = P_5'$, $p_3 + 2 = P_8$.

Here we study the solvability of the equation (1) in primes p_1, p_2, p_3, such that p_1+2, $p_2 + 2$, $p_3 + 2$ are almost–primes. We follow the approach of [9] putting emphasis on the examining of the main term where we apply some arguments of [1] (for the other details the reader may refer to [9]).

Our main result is the following

Theorem. *Suppose that $N \equiv 3 \pmod 6$ is a sufficiently large integer. Then there exist infinitely many solutions of the equation (1) in primes p_1, p_2, p_3, such that $p_1 + 2 = P_5$, $p_2 + 2 = P_5'$, $p_3 + 2 = P_8$.*

In fact, the proof yields that for some constant $c_0 > 0$ there are at least $c_0 N^2 (\log N)^{-6}$ triplets of primes p_1, p_2, p_3, satisfying (1) and such that for any prime factor p of $p_1 + 2$ or $p_2 + 2$ we have $p \geq N^{0.167}$ and for any prime factor p of $p_3 + 2$ we have $p \geq N^{0.116}$. Notice that if N is a sufficiently large odd integer, not satisfying the hypothesis of the Theorem, then for any solution of (1) we have $3 \mid p_1 p_2 p_3 (p_1+2)(p_2+2)(p_3+2)$. Therefore, by modifying slightly the given proof, we may obtain that for such N the equation (1) has infinitely many solutions in primes p_1, p_2, p_3, such that $p_1 + 2 = P_6$, $p_2 + 2 = P_6'$, $p_3 + 2 = P_9$. Here the extra prime factor in P_r is 3.

Recently H. Mikawa (unpublished result) used the theory of "well-factorable" functions and showed that the power of N in the quantity D_3 (for the definition see formulas (2)) can be taken to be equal to 4/9 instead of 1/3. This enables us to prove the Theorem with $p_3 + 2 = P_6$.

We should also mention that by applying the method of [9], D. I. Tolev [10] proved that if N is a sufficiently large integer satisfying the congruent condition $N \equiv 5 \pmod{24}$ then the equation

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 = N$$

has infinitely many sollutions in prime numbers p_1, p_2, p_3, p_4, p_5 such that each of the numbers $p_1 + 2$, $p_2 + 2$, $p_3 + 2$ and $p_4 + 2$ has at most 6 prime factors and $p_5 + 2$ has at most 7 prime factors.

§2. Notations.

Let N be a sufficiently large integer, such that $N \equiv 3 \pmod 6$ and α_1, α_2, α_3 – constants satisfying $0 < \alpha_1$, $\alpha_2 < 1/4$, $0 < \alpha_3 < 1/6$, which we shall specify later.

We put

$$\begin{gathered} z_i = N^{\alpha_i}, \quad i = 1, 2, 3, \qquad z_0 = (\log N)^{1000}, \qquad D_0 = \exp\left((\log N)^{0.6}\right), \\ D_1 = D_2 = N^{1/2} \exp\left(-2(\log N)^{0.6}\right), \quad D_3 = N^{1/3} \exp\left(-2(\log N)^{0.6}\right), \\ P(z_0) = \prod_{2<p<z_0} p, \qquad P(z_0, z_i) = \prod_{z_0 \leq p < z_i} p, \qquad i = 1, 2, 3. \end{gathered} \tag{2}$$

Letters m, n, d, l, k, h, δ, ν, t, ρ denote integers; p, p_1, p_2,...– prime numbers. As usual $\mu(n)$, $\varphi(n)$ and $\tau(n)$ denote Möbius' function, Euler's function and the number of positive divisors of n, respectively; $(m_1, ..., m_k)$ and $[m_1, ..., m_k]$ denote the greatest common divisor and the least common multiple of m_1,...,m_k. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n(k)$. The notation $p^\nu \| n$ means that $p^\nu | n$ and $p^{\nu+1} \nmid n$. For positive A and B we write $A \asymp B$ instead of $A \ll B \ll A$.

For squarefree odd integers k_1, k_2, k_3 and prime p we denote

$$I_{k_1,k_2,k_3}(N) = \sum_{\substack{p_1+p_2+p_3=N \\ p_i+2\equiv 0(k_i)\ ,\ i=1,2,3}} \log p_1 \log p_2 \log p_3\ , \tag{3}$$

$$h_{k_1,k_2,k_3}(p) = \begin{cases} 1/(p-1)^3 & \text{if } p \not| k_1k_2k_3,\ \ p\not| N\,; \\ -1/(p-1)^2 & \text{if } p \not| k_1k_2k_3,\ \ p\,|\,N\,; \\ -1/(p-1)^2 & \text{if } p \,\|k_1k_2k_3,\ \ p\not| N+2\,; \\ 1/(p-1) & \text{if } p \,\|k_1k_2k_3,\ \ p\,|\,N+2\,; \\ 1/(p-1) & \text{if } p^2\|k_1k_2k_3,\ \ p\not| N+4\,; \\ -1 & \text{if } p^2\|k_1k_2k_3,\ \ p\,|\,N+4\,; \\ -1 & \text{if } p^3\|k_1k_2k_3,\ \ p\not| N+6\,; \\ p-1 & \text{if } p^3\|k_1k_2k_3,\ \ p\,|\,N+6\,; \end{cases} \tag{4}$$

$$\omega(k_1,k_2,k_3) = \prod_{p|k_1k_2k_3} \frac{1+h_{k_1,k_2,k_3}(p)}{1+h_{1,1,1}(p)}\,, \qquad \Omega(k_1,k_2,k_3) = \frac{\omega(k_1,k_2,k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)}\,, \tag{5}$$

$$\mathfrak{S}(N) = \prod_{p\not| N}\Big(1+\frac{1}{(p-1)^3}\Big)\prod_{p|N}\Big(1-\frac{1}{(p-1)^2}\Big)\,. \tag{6}$$

§3. Outline of the proof.

Consider the sum

$$\Gamma = \sum_{\substack{p_1+p_2+p_3=N \\ (p_i+2,P(z_i))=1\ ,\ i=1,2,3}} \log p_1 \log p_2 \log p_3\,.$$

Any non-trivial estimate from below of Γ implies the solvability of (1) in primes, such that $p_i+2=P_{h_i}$, $h_i=[\alpha_i^{-1}]$, $i=1,2,3$. We see that

$$\Gamma = \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3\ \Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6\ ,$$

where

$$\Lambda_i = \begin{cases} \displaystyle\sum_{d|(p_i+2,P(z_0,z_i))} \mu(d) & \text{for } \ i=1,2,3\ , \\ \displaystyle\sum_{d|(p_{i-3}+2,P(z_0))} \mu(d) & \text{for } \ i=4,5,6\,. \end{cases}$$

Denote

$$\Lambda_i^{\pm} = \begin{cases} \displaystyle\sum_{d|(p_i+2,P(z_0,z_i))} \lambda_i^{\pm}(d) & \text{for } \ i=1,2,3\ , \\ \displaystyle\sum_{d|(p_{i-3}+2,P(z_0))} \lambda_0^{\pm}(d) & \text{for } \ i=4,5,6\,; \end{cases} \tag{7}$$

where $\lambda_i^{\pm}(d)$ are the Rosser's weights of order D_i , $0 \le i \le 3$ (see Iwaniec [6], [7]). In particular, we have

$$|\lambda_i^{\pm}(d)| \le 1 \, , \qquad \lambda_i^{\pm}(d) = 0 \ \text{ for } \ d \ge D_i \, , \qquad 0 \le i \le 3 \, . \tag{8}$$

We find that

$$\Lambda_i^- \le \Lambda_i \le \Lambda_i^+ \, , \qquad 1 \le i \le 6$$

(for the proof see [7]). Consequently we may apply Lemma 3 of [9], which is the analogue of Lemma 13 of [1], and we get

$$\Gamma \ge \Gamma_0 \, , \tag{9}$$

where

$$\Gamma_0 = \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 \; (\Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^+ +$$
$$+\Lambda_1^+ \Lambda_2^- \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^+ + \cdots + \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^- - 5\Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^+) \, .$$

We use (3), (7) and change the order of summation to obtain

$$\Gamma_0 = \sum_{\substack{d_i | P(z_0,z_i) \, , \ i=1,2,3 \\ \delta_i | P(z_0) \, , \ i=1,2,3}} \kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3) \; I_{d_1\delta_1, d_2\delta_2, d_3\delta_3}(N) \, , \tag{10}$$

where

$$\begin{aligned} \kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3) &= \lambda_1^-(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3) \\ &+ \ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \ \lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^-(\delta_3) \\ &- \ 5\lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3) \, . \end{aligned} \tag{11}$$

By applying the Hardy–Littlewood circle method we find an asymptotic formula for the sum $I_{k_1,k_2,k_3}(N)$ which we substitute in (10). Proceeding in the same way as in Lemma 11 and Lemma 13 of [9] we derive

$$\Gamma_0 = \tfrac{1}{2} \, N^2 \, \mathfrak{S}(N) W + \mathcal{O}\left(\frac{N^2}{\log^4 N}\right) \, , \tag{12}$$

where $\mathfrak{S}(N)$ is defined by (6) and

$$W = \sum_{\substack{d_i | P(z_0,z_i) \, , \ i=1,2,3 \\ \delta_i | P(z_0) \, , \ i=1,2,3}} \kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3) \; \Omega(d_1\delta_1, d_2\delta_2, d_3\delta_3) \, . \tag{13}$$

By (4) it is obvious that for squarefree odd integers k_1, k_2, k_3 we have

$$h_{k_1,k_2,k_3}(p) = \begin{cases} h_{1,1,1}(p) & \text{if} \ \ p \nmid k_1k_2k_3 \, ; \\ h_{p,1,1}(p) & \text{if} \ \ p \, \| k_1k_2k_3 \, ; \\ h_{p,p,1}(p) & \text{if} \ \ p^2 \| k_1k_2k_3 \, ; \\ h_{p,p,p}(p) & \text{if} \ \ p^3 \| k_1k_2k_3 \, . \end{cases} \tag{14}$$

Define

$$\omega_1(p) = \omega(p,1,1)\,, \quad \omega_2(p) = \omega(p,p,1)\,, \quad \omega_3(p) = \omega(p,p,p)\,. \tag{15}$$

It is clear that if $p > 2$ then

$$\omega_1(p) = \begin{cases} 1 & \text{if } \ p\,|\,N\,; \\ \frac{(p-1)^2}{p^2-3p+3} & \text{if } \ p\,|\,N+2\,; \\ \frac{(p-1)(p-2)}{p^2-3p+3} & \text{if } \ p\not|N(N+2)\,; \end{cases} \qquad \omega_2(p) = \begin{cases} \frac{p-1}{p-2} & \text{if } \ p\,|\,N\,; \\ 0 & \text{if } \ p\,|\,N+4\,; \\ \frac{(p-1)^2}{p^2-3p+3} & \text{if } \ p\not|N(N+4)\,; \end{cases} \tag{16}$$

$$\omega_3(p) = \begin{cases} 4 & \text{if } \ p=3\,; \\ \frac{(p-1)^3}{p^2-3p+3} & \text{if } \ p\,|N+6, \quad p>3\,; \\ 0 & \text{if } \ p\not|N+6\,. \end{cases}$$

By (4), (5), (14), (15) we get

$$\omega(k_1,k_2,k_3) = \prod_{\substack{p^\nu \| k_1k_2k_3 \\ 1\le\nu\le3}} \omega_\nu(p)\,. \tag{17}$$

The next statement is the analogue of Lemma 12 of [1]. The Lemma follows easily from (16), (17).

Lemma 1. *For squarefree odd k, let*

$$\omega^*(k) = \prod_{p|k} \omega_1(p)\,.$$

If k_1, k_2, k_3 is a triplet of integers, we put $k_{1,2} = (k_1,k_2)$, $k_{1,3} = (k_1,k_3)$, $k_{2,3} = (k_2,k_3)$. Then

(i) there exists a function g of the three variables $k_{i,j}$, such that for any squarefree odd k_1, k_2, k_3 we have

$$\omega(k_1,k_2,k_3) = \omega^*(k_1)\,\omega^*(k_2)\,\omega^*(k_3)\,g(k_{1,2},k_{1,3},k_{2,3})$$

and

$$g(k_{1,2},k_{1,3},k_{2,3}) \le 10\,(\max k_{i,j})^{10}\,;$$

(ii) for any squarefree odd k_1, k_2, k_3 we have the inequality

$$\omega(k_1,k_2,k_3) \le 10\,\tilde\omega(k_1)\,\tilde\omega(k_2)\,\tilde\omega(k_3)\,,$$

where $\tilde{\omega}(m)$ is the multiplicative function defined on squarefree odd m by

$$\tilde{\omega}(p) = \begin{cases} 2 & \text{if } p\nmid N+6\,; \\ 2p^{1/3} & \text{if } p|N+6\,. \end{cases}$$

Suppose that the integers $d_1, d_2, d_3, \delta_1, \delta_2, \delta_3$ satisfy the conditions imposed in (13). Using (5) and (17) we easily get

$$\Omega(d_1\delta_1, d_2\delta_2, d_3\delta_3) = \Omega(d_1, d_2, d_3)\ \Omega(\delta_1, \delta_2, \delta_3)\,.$$

Note that $\Omega(\delta_1, \delta_2, \delta_3)$ is a symmetrical function with respect to δ_1, δ_2, δ_3. Hence, we obtain by (11), (13)

$$W = \sum_{i=1}^{6} L_i H_i \ -\ 5L_7 H_7\,, \tag{18}$$

where

$$\begin{aligned}
L_1 &= \sum_{d_i|P(z_0,z_i)\,,\ i=1,2,3} \lambda_1^-(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\Omega(d_1,d_2,d_3)\,,\\
L_2 &= \sum_{d_i|P(z_0,z_i)\,,\ i=1,2,3} \lambda_1^+(d_1)\lambda_2^-(d_2)\lambda_3^+(d_3)\Omega(d_1,d_2,d_3)\,,\\
L_3 &= \sum_{d_i|P(z_0,z_i)\,,\ i=1,2,3} \lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^-(d_3)\Omega(d_1,d_2,d_3)\,,\\
L_4 = L_5 = L_6 = L_7 &= \sum_{d_i|P(z_0,z_i)\,,\ i=1,2,3} \lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\Omega(d_1,d_2,d_3)\,,\\
H_1 = H_2 = H_3 = H_7 &= \sum_{\delta_i|P(z_0)\,,\ i=1,2,3} \lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3)\Omega(\delta_1,\delta_2,\delta_3)\,,\\
H_4 = H_5 = H_6 &= \sum_{\delta_i|P(z_0)\,,\ i=1,2,3} \lambda_0^-(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3)\Omega(\delta_1,\delta_2,\delta_3)\,.
\end{aligned}$$

It is easy to prove the following

Lemma 2. *Suppose that $\phi(n_1, n_2, n_3)$ is a function defined on the set of integers and such that for any two triplets n_1, n_2, n_3 and l_1, l_2, l_3, satisfying $(n_1n_2n_3, l_1l_2l_3) = 1$, we have $\phi(n_1l_1, n_2l_2, n_3l_3) = \phi(n_1, n_2, n_3)\ \phi(l_1, l_2, l_3)$. Then the function*

$$\Phi(n) = \sum_{d_1,d_2,d_3|n} \phi(d_1, d_2, d_3)$$

is multiplicative.

Applying Lemma 1 and Lemma 2 we find asymptotic formulas for the sums H_i. Define

$$H^{(\mu)} = \sum_{\delta_i|P(z_0)\,,\ i=1,2,3} \mu(\delta_1)\ \mu(\delta_2)\ \mu(\delta_3)\ \Omega(\delta_1, \delta_2, \delta_3)\,. \tag{19}$$

Lemma 3. *We have*

$$H_i = H^{(\mu)} + \mathcal{O}((\log N)^{-10}) , \qquad 1 \le i \le 7 ,$$

and

$$H^{(\mu)} \asymp (\log z_0)^{-3} . \tag{20}$$

Now we are able to estimate from below the quantity W, defined by (18). We put

$$\mathcal{F}(z_0, z_i) = \prod_{z_0 \le p < z_i} \left(1 - \frac{\omega_1(p)}{p-1}\right) , \qquad s_i = \frac{\log D_i}{\log z_i} , \qquad i = 1, 2, 3 , \tag{21}$$

where $\omega_1(p)$ is defined by (16). Suppose that $c^* > 0$ is an absolute constant and let θ_i , s_i , $i = 1, 2, 3$ satisfy

$$\theta_1 + \theta_2 + \theta_3 = 1 , \qquad \theta_i > 0 , \qquad f(s_i) - 2\theta_i F(s_i) > c^* , \qquad i = 1, 2, 3 ,$$

where f and F are the functions of the linear sieve. Following the arguments in the proof of Lemma 15 of [9] it is easy to establish that

$$W \ge H^{(\mu)} \prod_{j=1}^{3} \mathcal{F}(z_0, z_j) \left(\sum_{i=1}^{3} \Big(f(s_i) - 2\theta_i F(s_i) \Big) + \mathcal{O}\left((\log N)^{-1/3} \right) \right) . \tag{22}$$

Finally, we choose

$$\alpha_1 = \alpha_2 = 0.167 , \qquad \alpha_3 = 0.116 , \qquad \theta_1 = \theta_2 = 0.345 , \qquad \theta_3 = 0.31$$

and compute that for sufficiently large N we have

$$f(s_i) - 2\theta_i F(s_i) > 10^{-5} , \qquad i = 1, 2, 3 . \tag{23}$$

Therefore, using (2), (20)–(23) we get

$$W \gg \frac{1}{\log^3 N} .$$

The last estimate and (9), (12) imply

$$\Gamma \gg \frac{N^2}{\log^3 N} ,$$

which suffices to complete the proof of the Theorem.

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